Multiple analogical proportions

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based on a paper to appear in *AI Communications*

Contents

- Short background on analogical proportions
- Analogical proportions in an old treatise
- Multiple analogical proportions

Analogical proportions

- " *a* is to *b* as *c* is to *d*"
  a differs from *b* as *c differs* from *d*
  and *b differs* from *a* as *d differs* from *c*".
- $a : b :: c : d \triangleq$
  $((a \wedge \neg b) \equiv (c \wedge \neg d)) \wedge ((\neg a \wedge b) \equiv (\neg c \wedge d))$
  it uses dissimilarity indicators only
- $a : b :: c : d$ satisfies the key properties of an analogical proportion, namely
  - reflexivity: $a : b : a : b$
  - symmetry: $a : b :: c : d \Rightarrow c : d :: a : b$
  - central permutation: $a : b :: c : d \Rightarrow a : c :: b : d$
  - also satisfies $a : a :: b : b$
    and *external permutation* $a : b :: c : d \Rightarrow d : b :: c : a$

Analogical proportion truth table

Boolean patterns making analogical proportion $a : b :: c : d$ true

a	b	c	d
0	0	0	0
1	1	1	1
0	0	1	1
1	1	0	0
0	1	0	1
1	0	1	0

- analogical proportion is transitive:

$$(a : b :: c : d) \wedge (c : d :: e : f) \Rightarrow a : b :: e : f$$

- multiple-valued logic extensions

Arithmetic and geometric proportions

Numerical proportions:

- Arithmetic proportion: $a - b = c - d$
- *compatible* with $a : b :: c : d$ **but** $a - b \in \{-1, 0, 1\}$
- Geometric proportion: $\frac{a}{b} = \frac{c}{d}$
- These two basic proportions can be exchanged by logarithmic / exponential transformations:
  if $\frac{a}{b} = \frac{c}{d}$ then we have $\ln(a) - \ln(b) = \ln(c) - \ln(d)$
  if $a - b = c - d$ then we have $\frac{e^a}{e^b} = \frac{e^c}{e^d}$.

Analogical proportions between vectors

- Items are represented by *vectors* of Boolean values $\vec{a} = (a_1, ..., a_n)$
- $\vec{a} : \vec{b} :: \vec{c} : \vec{d}$ iff $\forall i \in [1, n],\ a_i : b_i :: c_i : d_i$
- Pairing pairs (a, b) and (c, d)

	$\mathcal{A}_1$	...	$\mathcal{A}_{i-1}$	$\mathcal{A}_i$	...	$\mathcal{A}_{j-1}$	$\mathcal{A}_j$	...	$\mathcal{A}_{k-1}$	$\mathcal{A}_k$	...	$\mathcal{A}_{r-1}$	$\mathcal{A}_r$	...	$\mathcal{A}_{s-1}$	$\mathcal{A}_s$	...	$\mathcal{A}_n$
a	1	...	1	0	...	0	1	...	1	0	...	0	1	...	1	0	...	0
b	1	...	1	0	...	0	1	...	1	0	...	0	0	...	0	1	...	1
c	1	...	1	0	...	0	0	...	0	1	...	1	1	...	1	0	...	0
d	1	...	1	0	...	0	0	...	0	1	...	1	0	...	0	1	...	1

On attributes $\mathcal{A}_1$ to $\mathcal{A}_{r-1}$ $\vec{a}$ and $\vec{b}$ *agree* and $\vec{c}$ and $\vec{d}$ *agree* as well.
It contrasts with attributes $\mathcal{A}_r$ to $\mathcal{A}_n$, for which we can see that $\vec{a}$ *differs* from $\vec{b}$ as $\vec{c}$ *differs* from $\vec{d}$ (and vice-versa)

Analogical proportions: A machinery for comparing items

- Analogical proportions for Boolean vectors (it extends to *nominal* values)

$\vec{a} : \vec{b} :: \vec{c} : \vec{d}$ holds if and only if

$\mathcal{A}^{=}_{\vec{a},\vec{b}} = \mathcal{A}^{=}_{\vec{c},\vec{d}}$, $\mathcal{D}^{01}_{\vec{a},\vec{b}} = \mathcal{D}^{01}_{\vec{c},\vec{d}}$ and $\mathcal{D}^{10}_{\vec{a},\vec{b}} = \mathcal{D}^{10}_{\vec{c},\vec{d}}$

where

$\mathcal{A}^{=}_{\vec{a},\vec{b}} = \{i \mid a_i = b_i\}$

$\mathcal{D}^{01}_{\vec{a},\vec{b}} = \{i \mid a_i = 0, b_i = 1\}$ and

$\mathcal{D}^{10}_{\vec{a},\vec{b}} = \{i \mid a_i = 1, b_i = 0\}$

- $\vec{d}$ *can be computed* from $\vec{a}$, $\vec{b}$ and $\vec{c}$
- the 4 vectors are in general all different

Pairing pairs (a, d) *and* (b, c)

- $\vec{a} : \vec{d} :: \vec{b} : \vec{c}$ *does not hold*: see attributes $\mathcal{A}_s$ to $\mathcal{A}_n$

	$\mathcal{A}_1$	...	$\mathcal{A}_{i-1}$	$\mathcal{A}_i$	...	$\mathcal{A}_{j-1}$	$\mathcal{A}_j$	...	$\mathcal{A}_{k-1}$	$\mathcal{A}_k$	...	$\mathcal{A}_{r-1}$	$\mathcal{A}_r$	...	$\mathcal{A}_{s-1}$	$\mathcal{A}_s$	...	$\mathcal{A}_n$
a	1	...	1	0	...	0	1	...	1	0	...	0	1	...	1	0	...	0
d	1	...	1	0	...	0	0	...	0	1	...	1	0	...	0	1	...	1
b	1	...	1	0	...	0	1	...	1	0	...	0	0	...	0	1	...	1
c	1	...	1	0	...	0	0	...	0	1	...	1	1	...	1	0	...	0

$a : b :: c : d = ((a \wedge d) \equiv (b \wedge c)) \wedge ((\neg a \wedge \neg d) \equiv (\neg b \wedge \neg c))$
or equivalently
$a : b :: c : d = ((a \wedge d) \equiv (b \wedge c)) \wedge ((a \vee d) \equiv (b \vee c))$

a matter of similarity

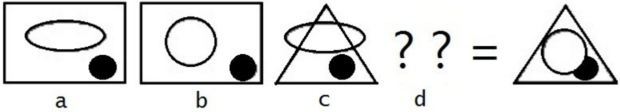
Analogical inference

- Equation $a : b :: c : x$ may not have a solution in $\mathbb{B}$
  neither $0 : 1 :: 1 : x$ nor $1 : 0 :: 0 : x$ have a solution
- when it exists (iff $(a \equiv b) \vee (a \equiv c)$ holds) it is unique
- $x = c \equiv (a \equiv b)$
- Applies to Boolean vectors: look for
  $\vec{x} = (x_1, \cdots, x_n)$ s.t. $\vec{a} : \vec{b} :: \vec{c} : \vec{x}$ holds:
  $\Rightarrow$ n equations $a_i : b_i :: c_i : x_i$

analogical proportion solving process is *creative*

$\vec{x} \neq \vec{a}$, $\vec{x} \neq \vec{b}$, $\vec{x} \neq \vec{c}$

Puzzles



Th. Evans (1963)

Example encoded with 5 Boolean predicates

hasRectangle(*hR*), *hasBlackDot*(*hBD*), *hasTriangle*(*hT*)

hasCircle(*hC*), *hasEllipse*(*hE*) (in that order)

	hR	*hBD*	*hT*	*hC*	*hE*
a	1	1	0	0	1
b	1	1	0	1	0
c	0	1	1	0	1
x	?	?	?	?	?

General analogical inference

- $$\frac{\forall i \in \{1, ..., p\}, \quad a_i : b_i :: c_i : d_i \text{ holds}}{\forall j \in \{p+1, ..., n\}, \quad a_j : b_j :: c_j : d_j \text{ holds}}$$

  (Stroppa, Yvon, 2005)
- analogical reasoning amounts to finding completely informed triples $(\vec{a}, \vec{b}, \vec{c})$ suitable for inferring the missing value(s) of an incompletely informed item $(\vec{d})$
- if *several triples* leading to distinct conclusions a *voting* procedure may be used
- extends to gradual analogical proportions

Classification

M. Bounhas, H. Prade, G. Richard. Analogy-based classifiers for nominal or numerical data. IJAR 91, 36-55, 2017

- direct application of general inference principle
- one has to predict a class $cl(\vec{x})$ (viewed as a nominal attribute) for a new item $\vec{x}$
- successively applied to Boolean, nominal and numerical attributes
- analogical classifiers always give exact predictions when the classification process is governed by an affine Boolean function (which includes x-or functions) and *only in this case*

does not prevent to get good results in other cases (as observed in practice)

Difference with case-based reasoning

Analogical proportion-based inference $\neq$ CBR:

It takes advantage of **triples**
for extrapolating conclusions,

while Case-Based Reasoning exploits the similarity of the new case
with stored cases considered one by one

An old treatise

- Gaspard Monge. *Traité Élémentaire de Statique, à l'usage des Écoles de la Marine* (8 éd. de 1788 à 1846)
- THEOREM. Fig. 14. If three forces P, Q, R, be represented in intensity and direction by the three sides AB, AC, AD, adjacent to the same angle of a parallelopipedon $ABFEGD$, so that

$$P : Q : R :: AB : AC : AD,$$

their resultant S will be represented in intensity and direction by the diagonal AE of the parallelopipedon adjacent to the same angle, and we shall have

$$P : Q : R : S :: AB : AC : AD : AS.$$

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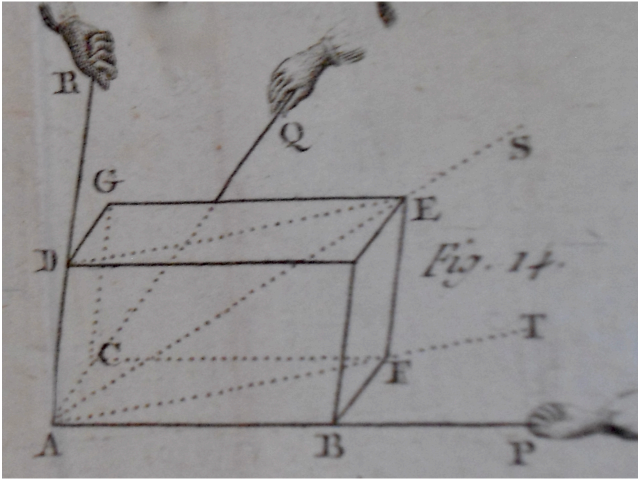

Figure: Fig. 14 in Gaspard Monge's book

A vectorial view of Boolean analogical proportions - 1

Four items *a*, *b*, *c*, *d* represented by Boolean vectors $\overrightarrow{a}$, $\overrightarrow{b}$, $\overrightarrow{c}$, $\overrightarrow{d}$ on a set of *n* features form an analogical proportion componentwise if and only if *abdc* is a parallelogram in $\mathbb{R}^n$

The `man` : `king` :: `woman` : `queen` example

	sexM	*sexF*	*power position*	*ordinary position*	*human*	*god*
Man ($\overrightarrow{M}$)	1	0	0	1	1	0
King ($\overrightarrow{K}$)	1	0	1	0	1	0
Woman ($\overrightarrow{W}$)	0	1	0	1	1	0
Queen ($\overrightarrow{Q}$)	0	1	1	0	1	0

$$\overrightarrow{MK} = \overrightarrow{K} - \overrightarrow{M} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} = \overrightarrow{WQ} = \overrightarrow{Q} - \overrightarrow{W} = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

A vectorial view of Boolean analogical proportions - 2

- If four Boolean vectors make an analogical proportion $\vec{a} : \vec{b} :: \vec{c} : \vec{d}$ then $\vec{d} = \vec{a} + \overrightarrow{ab} + \overrightarrow{ac}$

- $$\vec{Q} = \vec{M} + \overrightarrow{MQ} = \vec{M} + \overrightarrow{MK} + \overrightarrow{MW}.$$

- Conversely, $\vec{d} = \vec{a} + \overrightarrow{ab} + \overrightarrow{ac}$ can be rewritten as $\vec{d} - \vec{a} = \overrightarrow{ad} = \overrightarrow{ab} + \overrightarrow{ac}$. So *abdc* is a parallelogram
  Then $\vec{a} : \vec{b} :: \vec{c} : \vec{d}$ holds true,
  provided that $\nexists i$ such that $a_i = 0$ and $b_i = c_i = 1$, or such that $a_i = 1$ and $b_i = c_i = 0$, in order to guarantee the existence of a Boolean solution $\vec{d}$

Multiple analogical proportions

- in Gaspard Monge 's book
  $x_1 : x_2 : \cdots : x_n :: y_1 : y_2 : \cdots : y_n$,
  is to be understood as $\frac{x_1}{y_1} = \frac{x_2}{y_2} = \cdots = \frac{x_n}{y_n}$,
- from $a : b : c :: x : y : z$ Monge draws the numerical equalities $b = \frac{ay}{x}$ and $c = \frac{az}{x}$. From which, one can easily conclude $\frac{b}{c} = \frac{y}{z}$ (by eliminating $\frac{a}{x}$)
- Counterpart for Boolean analogical proportions?
- See $a : b : c :: x : y : z$
  as the conjunction of $a : b :: x : y$ and $a : c :: x : z$
- by applying central permutation, symmetry and transitivity $b : c :: y : z$ follows

Boolean valuations and postulates

000111
111000
000000
001001
010010
011011
100100
101101
110110
111111

Table: Valid valuations for $a : b : c :: x : y : z$

2 postulates $a : a : a :: x : x : x$ and $a : b : c :: a : b : c$

symmetry still holds

Inference and triangulation: Analogy solving problem

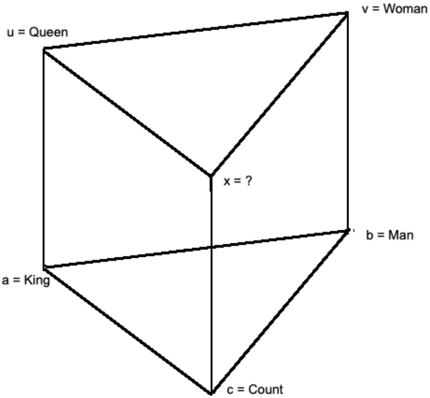

Figure: Triangulation example

Since the proportion `King:Man::Queen:Woman` holds,
the two solutions *x* should be the same

Triple analogical proportion

- In the *numerical* case, $a : b : c : d :: x : y : z : t$ is understood as $b = \frac{ay}{x}$, $c = \frac{az}{x}$ and $d = \frac{at}{x}$
- in the *Boolean* setting $a : b : c : d :: x : y : z : t$ can be defined by the conjunction of 3 proportions: $a : b :: x : y$, $a : c :: x : z$ and $a : d :: x : t$ from which one can derive *3 other proportions* $b : c :: y : z$, $b : d :: y : t$, and $c : d :: z : t$, using transitivity
- stating that $a_1 : a_2 : \cdots : a_n :: x_1 : x_2 : \cdots : x_n$ holds, establishes a parallel between two situations, one described by the a_i's, the other described by the x_i's, with a one to one correspondence

Triple analogical proportion

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Example - 1

dog:pig:wolf:wild boar ::
puppy:piglet:wolf cub:young wild boar

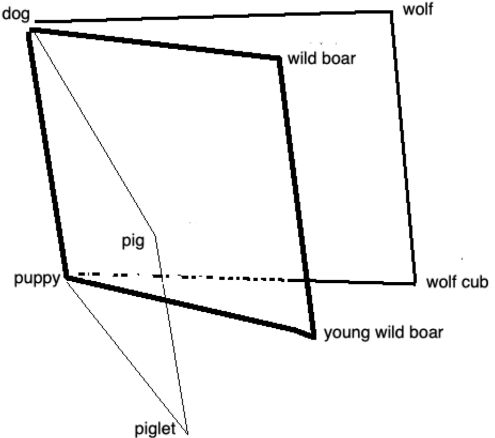

Figure: Animals and their offsprings.

Example - 2

dog:pig:piglet:puppy ::
wolf:wild boar:young wild boar:wolf cub

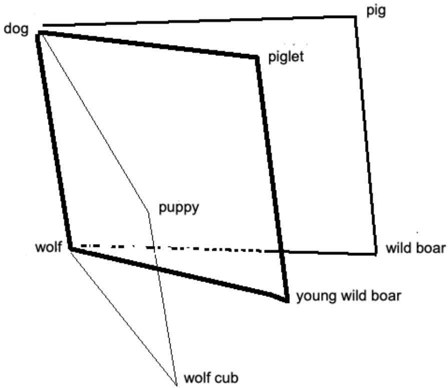

Figure: Domestic animals and corresponding wild animals.

dog:puppy:wolf:wolf cub ::
pig:piglet:wild boar:young wild boar

Analogical proportions are easy to obtain

- by taking pairs of mutually exclusive Boolean properties ($p \wedge p' = \bot$, $q \wedge q' = \bot$, $r \wedge r' = \bot$),
  and considering four items a, a', b, b'
  respectively described on the 6 properties (p, q, r, r', q', p') by

	p	q	r	r'	q'	p'
a	1	1	1	0	0	0
a'	1	1	0	1	0	0
b	1	0	1	0	1	0
b'	1	0	0	1	1	0

$a : a' :: b : b'$ and a, a', b, b' make a kind of square of opposition (not the traditional one!)

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b'	1	0	0	1	1	0

  $a : a' :: b : b'$ and a, a', b, b' make a kind of square of opposition
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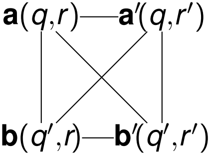
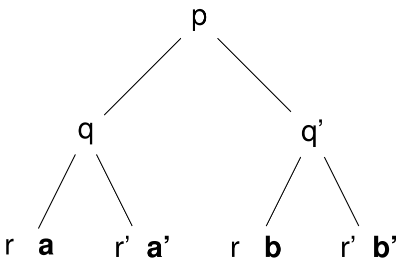
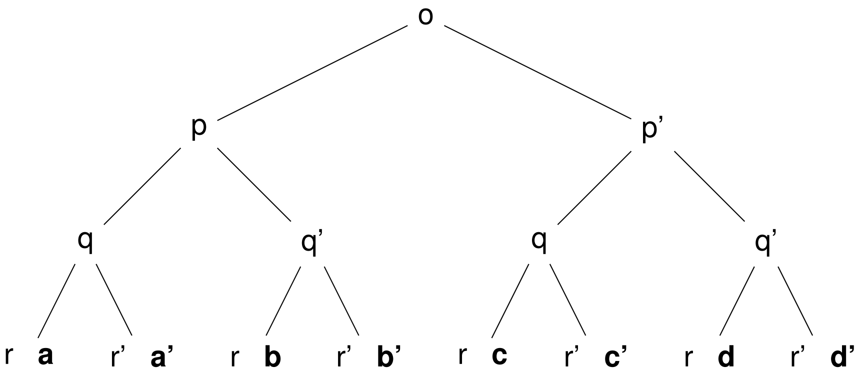


An equivalent binary (classification) tree



Introducing one more level in the tree

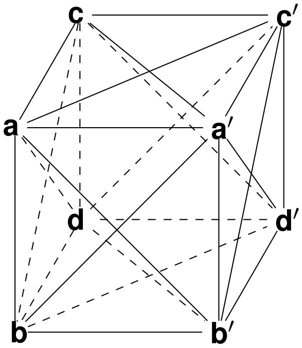


From the square to the cube

Table 1	*o*(animal)	*p*(canid)	*q*(tame)	*r*(young)	*r'*(adult)	*q'*(wild)	*p'*(suidae)	*o'*(plant)
a *puppy*	1	1	1	1	0	0	0	0
a′ *dog*	1	1	1	0	1	0	0	0
b *wolfcub*	1	1	0	1	0	1	0	0
b′ *wolf*	1	1	0	0	1	1	0	0
c *piglet*	1	0	1	1	0	0	1	0
c′ *pig*	1	0	1	0	1	0	1	0
d *yg.wd.bo.*	1	0	0	1	0	1	1	0
d′ *wildboar*	1	0	0	0	1	1	1	0

Analogical cube

Parallel facets, parallel edges are in opposition



Clearly we can then generate *hypercubes* by introducing more dichotomies

Concluding remarks

- *Analogical proportions do not come in isolation*
- use of analogy-based explanations in machine learning
- interfacing symbolic (high level features) and numerical (embeddings) representations